Realization and Discretization of Asymptotically Stable Homogeneous Systems

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***Abstract*—Sufficient conditions for the existence and convergence to zero of numeric approximations to solutions of asymptotically stable homogeneous systems are obtained for the explicit and implicit Euler integration schemes. It is shown that the explicit Euler method has certain drawbacks for the global approximation of homogeneous systems with non-zero degrees, whereas the implicit Euler scheme ensures convergence of the approximating solutions to zero. Properties of absolute and relative errors of the respective discretizations are investigated.**

I. Introduction

The problems of stability/performance analysis and control design for continuous-time dynamical systems are very popular and important nowadays. If the system model is linear, then the theory is very-well developed and plenty of approaches exist to solve these problems. In many cases, due to inherited nonlinearity of the plant dynamics, or due to complex quality restrictions imposed on the controlled system, the closed-loop stays nonlinear. Analysis and design methods for such systems are demanded in many applications and are quickly developing the last decades.

Homogeneous dynamical systems become popular since they take an intermediate place between linear and nonlinear systems [1]. They possess some properties of linear ones (*e.g.* the scalability of trajectories), while being described by essentially nonlinear differential equations, which add such qualities as robustness to measurement noises, exogenous disturbances and delays [2], or an increased rate of convergence to the goal invariant set.

Frequently for a continuous-time system, after the analysis or design have been performed, for verification or implementation, the system solutions have to be calculated in a computer or in a digital controller (*e.g.* in a state observer). For these purposes different numerical approximation methods and discretization schemes are used [3], [4]. The Euler method is a first-order numerical routine for solving ordinary differential equations with a given initial value and time step, which represents the most basic explicit/implicit method of numerical integration and is the simplest Runge-Kutta method. Since homogeneous systems theory is expanding, there are many control or estimation algorithms proposed recently, which possess an increased rate of convergence with respect to linear systems (finite-time or fixed-time convergences [1], [5]), that is why implementation and derivation of solutions for such homogeneous systems need an additional attention, which is the subject of the present research regarding the Euler method.

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The conditions of convergence and stability of the explicit and implicit Euler methods have been studied for linear systems (the notion of A-stability) [4], [6], or for particular classes of nonlinear systems [7], [8]. This paper continues a series of works devoted to application of various discretization schemes for approximation of solutions of homogeneous stable dynamical systems [9], [10] (given without proofs). In the present work, different conditions for the existence and convergence to zero of solutions for the explicit and implicit Euler integration schemes are obtained for homogeneous systems. It is shown that application of the explicit Euler method for the global approximation of solutions of homogeneous systems with non-zero degree is problematic (see also [11]), and the implicit Euler scheme has a better perspective (see also [12], [13], [14], [15]). However, it is worth to stress that the implicit Euler method has higher computational complexity than the explicit one. Several conditions are proposed, which guarantee existence and convergence to zero of approximations derived by the implicit Euler approach. Absolute and relative errors (closeness of the approximations to real solutions) for the explicit and implicit Euler integration schemes are investigated using the homogeneity theory (have not been considered previously in [9]).

The outline of this paper is as follows. The notation and preliminary results are introduced in sections II and III. Some basic properties and relations between solution approximations are studied in Section IV. The convergence and divergence conditions are established in Section V. Properties of relative and absolute errors of approximation of solutions of homogeneous systems by the explicit and implicit Euler methods are investigated in Section VI. Several simple illustrating examples are considered in Section VII.

II. Notation

- $\mathbb{R}_+ = \{x \in \mathbb{R} : x \geq 0\}$, where $\mathbb{R}$ is the set of real number.
- $|\cdot|$ denotes the absolute value in $\mathbb{R}$, $\|.\|$ denotes the Euclidean norm on $\mathbb{R}^n$.
- A continuous function $\alpha : \mathbb{R}_+ \to \mathbb{R}_+$ belongs to the class $\mathcal{K}$ if $\alpha(0) = 0$ and the function is strictly increasing. The function $\alpha : \mathbb{R}_+ \to \mathbb{R}_+$ belongs to the class $\mathcal{K}_\infty$ if $\alpha \in \mathcal{K}$ and it is increasing to infinity.
- The identity matrix of dimension $n \times n$ is denoted as I_n, and $\text{diag}\{r_i\}_{i=1}^n$ is a diagonal matrix with the elements on the main diagonal equal r_i.
- A sequence of integers $1, 2, ..., n$ is denoted by $\overline{1, n}$.

III. Preliminaries

In this work the following nonlinear system is considered:

$$\dot{x}(t) = f(x(t)),\ t \geq 0, \tag{1}$$

where $x(t) \in \mathbb{R}^n$ is the state, $f : \mathbb{R}^n \to \mathbb{R}^n$ ensures forward existence and uniqueness of the system solutions at least locally, $f(0) = 0$. For an initial condition $x_0 \in \mathbb{R}^n$ define the corresponding solution by $X(t, x_0)$ for any $t \geq 0$ for which the solution exists. If f is discontinuous, then the solutions are understood in the Filippov's sense [16].

Following [17], [18], [5], let $\Omega \subset \mathbb{R}^n$ be an open set, $0 \in \Omega$.

Definition 1. At the steady state $x = 0$ the system (1) is said to be

(a) *stable* if for any $x_0 \in \Omega$ the solution $X(t, x_0)$ is defined for all $t \geq 0$, and for any $\epsilon > 0$ there is $\delta > 0$ such that for any $x_0 \in \Omega$, if $\|x_0\| \leq \delta$ then $\|X(t, x_0)\| \leq \epsilon$ for all $t \geq 0$;

(b) *asymptotically stable* if it is stable and for any $\kappa > 0$ and $\epsilon > 0$ there exists $T(\kappa, \epsilon) \geq 0$ such that for any $x_0 \in \Omega$, if $\|x_0\| \leq \kappa$ then $\|X(t, x_0)\| \leq \epsilon$ for all $t \geq T(\kappa, \epsilon)$;

(c) *finite-time stable* if it is stable and *finite-time converging from* Ω, *i.e.* for any $x_0 \in \Omega$ there exists $0 \leq T < +\infty$ such that $X(t, x_0) = 0$ for all $t \geq T$. The function $T_0(x_0) = \inf\{T \geq 0 : X(t, x_0) = 0 \ \forall t \geq T\}$ is called the *settling time* of the system (1);

(d) *fixed-time stable* if it is finite-time stable and $\sup_{x_0 \in \Omega} T_0(x_0) < +\infty$.

The set Ω is called a *domain of stability/attraction*.

If $\Omega = \mathbb{R}^n$, then the corresponding properties are called *global* stability/asymptotic stability/finite-time/fixed-time stability of (1) at $x = 0$.

Similarly, the stability notions can be defined with respect to a compact invariant set, by replacing the distance to the origin in Definition 1 with the distance to the invariant set.

A. Weighted homogeneity

Following [19], [1], [20], for strictly positive numbers r_i, $i = \overline{1, n}$ called weights and $\lambda > 0$, define:

- the *vector of weights* $\mathbf{r} = (r_1, \ldots, r_n)^T$, $r_{\max} = \max_{1 \leq j \leq n} r_j$ and $r_{\min} = \min_{1 \leq j \leq n} r_j$;
- the *dilation matrix* function $\Lambda_r(\lambda) = diag\{\lambda^{r_i}\}_{i=1}^n$, note that $\forall x \in \mathbb{R}^n$ and $\forall \lambda > 0$ we have $\Lambda_r(\lambda)x = (\lambda^{r_1} x_1, \ldots, \lambda^{r_n} x_n)^T$;
- the $\mathbf{r}$*–homogeneous norm* $\|x\|_r = (\sum_{i=1}^n |x_i|^{\frac{\rho}{r_i}})^{\frac{1}{\rho}}$ for any $x \in \mathbb{R}^n$ and $\rho \geq r_{\max}$, it is not a norm in the standard sense, since the triangle inequality is not satisfied for $\| \cdot \|_r$, however there exist $\underline{\sigma}, \overline{\sigma} \in \mathcal{K}_\infty$ such that
$$\underline{\sigma}(\|x\|_r) \leq \|x\| \leq \overline{\sigma}(\|x\|_r) \quad \forall x \in \mathbb{R}^n;$$
- the *sphere* and the *ball in the homogeneous norm* $S_r(\rho) = \{x \in \mathbb{R}^n : \|x\|_r = \rho\}$ and $B_r(\rho) = \{x \in \mathbb{R}^n : \|x\|_r \leq \rho\}$ for $\rho \geq 0$.

Definition 2. A function $g : \mathbb{R}^n \to \mathbb{R}$ is $\mathbf{r}$*–homogeneous with degree* $\mu \in \mathbb{R}$ if $\forall x \in \mathbb{R}^n$ and $\forall \lambda > 0$ we have:
$$\lambda^{-\mu} g(\Lambda_r(\lambda)x) = g(x).$$

A vector field $f : \mathbb{R}^n \to \mathbb{R}^n$ is $\mathbf{r}$*–homogeneous with degree* $\nu \in \mathbb{R}$, with $\nu \geq -r_{\min}$ if $\forall x \in \mathbb{R}^n$ and $\forall \lambda > 0$ we have:
$$\lambda^{-\nu} \Lambda_r^{-1}(\lambda) f(\Lambda_r(\lambda)x) = f(x),$$
which is equivalent for i-th component of f being a $\mathbf{r}$–homogeneous function of degree $r_i + \nu$.

System (1) is $\mathbf{r}$–homogeneous of degree ν if the vector field f is $\mathbf{r}$–homogeneous of the degree ν.

Theorem 1. *[19], [21] For the system (1) with* $\mathbf{r}$*–homogeneous and continuous function* f *the following properties are equivalent:*

- *the system* (1) *is (locally) asymptotically stable*;
- *there exists a continuously differentiable* $\mathbf{r}$*–homogeneous Lyapunov function* $V : \mathbb{R}^n \to \mathbb{R}_+$ *such that*
$$\alpha_1(\|x\|) \leq V(x) \leq \alpha_2(\|x\|),$$
$$L_f V(x) = \frac{\partial V}{\partial x}(x) f(x) \leq -\alpha(\|x\|),$$
$$\lambda^{-\mu} V(\Lambda_r(\lambda)x) = V(x), \ \mu > r_{\max},$$
$\forall x \in \mathbb{R}^n$ *and* $\forall \lambda > 0$, *for some* $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$ *and* $\alpha \in \mathcal{K}$.

The requirement on continuity of the function f has been relaxed in [22] (the function V can still be selected smooth).

IV. Euler schemes

If it comes to approximate solution $X(t, x_0)$ of the system (1) for some initial state $x_0 \in \mathbb{R}^n$, then different discretization schemes are used, where the most popular one is the discretization obtained by applying the Euler method (explicit or implicit) [4]. To this end, select a discretization step $h > 0$, define a sequence of time instants $t_i = ih$ for $i = 0, 1, \ldots$, and denote by x_i an approximation of the solution $X(t_i, x_0)$ at the corresponding time instant (*i.e.* $x_i \simeq X(t_i, x_0)$ and $x_0 = x(t_0) = x(0)$), then the approximation x_{i+1} calculated in accordance with the explicit Euler method is given by [4]:
$$x_{i+1} = x_i + h f(x_i) \tag{2}$$
for $i = 0, 1, \ldots$, while the approximation calculated by the implicit Euler method comes from [4]:
$$x_{i+1} = x_i + h f(x_{i+1}) \tag{3}$$
for $i = 0, 1, \ldots$ It is a well-known fact that with $h \to 0$ both methods approach the real solution [4], *i.e.* $x_i \to X(t_i, x_0)$ in (2) and (3) with $h \to 0$ over any compact time interval (this issue is investigated in Section VI). In the sequel, the problem of convergence to zero of the approximations $\{x_i\}_{i=0}^\infty$ derived in (2) and (3) is studied for stable and homogeneous system (1):

Assumption 1. *Let (1) be* $\mathbf{r}$*–homogeneous with a degree* ν *and asymptotically stable.*

To proceed we need to establish some properties of solutions in (2) and (3).

A. Existence of approximations

Existence of some $x_{i+1} \in \mathbb{R}^n$ for any $x_i \in \mathbb{R}^n$ in the explicit case (2) is straightforward, but it is not the case of (3). From homogeneity property we can obtain the following result.

Proposition 1. *Let system (1) be* $\mathbf{r}$*–homogeneous with a degree* $\nu \neq 0$. *Let for any* $x_0 \in S_r(1)$ *and all* $h > 0$ *there exist a sequence* $\{x_i\}_{i=0}^\infty$ *obtained by (3) with initial state* x_0. *Then for any discretization step* $h' > 0$ *and for any* $y_0 \in \mathbb{R}^n$ *there exist a sequence* $\{y_i\}_{i=0}^\infty$ *generated by (3) with the step* h' *and the initial state* y_0.

Proof. Note that since $f(0) = 0$, then for $y_0 = 0$ always there is a sequence $y_i = 0$, $i \geq 0$ satisfying (2). Consider $y_0 \neq 0$.

First, for any $x_i \in S_r(1)$ denote $x_{i+1}^h \in \mathbb{R}^n$ as the corresponding solution of the equation (3) for any step $h > 0$:
$$x_{i+1}^h = x_i + h f(x_{i+1}^h),$$
which exists by the imposed restrictions. For any $y_i \in \mathbb{R}^n \backslash \{0\}$ ($\|y_i\|_r \neq 0$ and $\Lambda_r(\|y_i\|_r^{-1})$ is a diagonal invertible matrix) and any $h' > 0$, consider $x_i = \Lambda_r(\|y_i\|_r^{-1}) y_i$ with $x_i \in S_r(1)$ and $h = \|y_i\|_r^\nu h'$, then we have
$$\Lambda_r(\|y_i\|_r) x_{i+1}^h = \Lambda_r(\|y_i\|_r) x_i + h' f(\Lambda_r(\|y_i\|_r) x_{i+1}^h)$$
or
$$y_{i+1} = y_i + h' f(y_{i+1})$$
for $y_{i+1} = \Lambda_r(\|y_i\|_r) x_{i+1}^h$, which is a solution to (3) for y_i and the step h'. Thus, for any $y_i \in \mathbb{R}^n$ and step $h' > 0$ there exists a solution $y_{i+1} \in \mathbb{R}^n$ satisfying (3). Such an operation can be repeated iteratively to substantiate existence of sequences y_i, $i \geq 0$. □

Thus, if there exist sequences $\{x_i\}_{i=0}^{\infty}$ generated by (2) or (3) with initial state $x_0 \in S_r(1)$ for any $h > 0$, then some sequences $\{y_i\}_{i=0}^{\infty}$ will exist for any $y_0 \in \mathbb{R}^n$ and any step $h > 0$, but it is hard to make a conclusion about boundedness or convergence of these sequences $\{y_i\}_{i=0}^{\infty}$.

In general case, it is difficult to provide some simple conditions for existence and uniqueness of solution of the equation (3). Homogeneity simplifies some derivations.

Proposition 2. *If the function $f : \mathbb{R}^n \to \mathbb{R}^n$ is continuously differentiable outside the origin,* $\mathbf{r}$*–homogeneous of degree $\nu \neq 0$ and there exists $h_0 > 0$ such that*

$$\det\left(I_n - h_0 \frac{\partial f(x)}{\partial x}\right) \neq 0 \quad \forall x \in \mathbb{R}^n \setminus \{0\}, \tag{4}$$

then for $n \geq 2$ the equation (3) *has a solution with respect to $x_{i+1} \in \mathbb{R}^n$ for any $x_i \in \mathbb{R}^n$ and for any $h > 0$, additionally, for $n \geq 3$ the solution is unique.*

Proof. I. The function f is continuously differentiable in $\mathbb{R}^n \backslash \{0\}$, so it is continuous in $\mathbb{R}^n \backslash \{0\}$. By homogeneity we have $f(\Lambda_r(\lambda)x) = \lambda^\nu \Lambda_r(\lambda) f(x)$ for all $x \in S_r(1)$ and all $\lambda > 0$. Since $\nu > -r_{\min}$ and $r_i > 0$ then $\lambda^\nu \Lambda_r(\lambda) \to 0$ and $\Lambda_r(\lambda)x \to 0$ as $\lambda \to 0$. Hence, f is also continuous at zero.

II. Let us consider the continuous function $F : \mathbb{R}^n \to \mathbb{R}^n$ defined as $F(x) = x - h_0 f(x)$. The function F is radially unbounded, *i.e.* $\|F(x)\| \to \infty$ as $\|x\| \to \infty$. Indeed, for $x \neq 0$ we have $F(x) = x - h_0 f(x) = \Lambda_r(\|x\|_r)(\tilde{x} - h_0 \|x\|_r^\nu f(\tilde{x}))$, where $\tilde{x} = \Lambda_r^{-1}(\|x\|_r)x \in S_r(1)$. Since $\nu \neq 0$ then $\|h_0 \|x\|_r^\nu f(\tilde{x}))\| \to 0$ as $\|x\| \to \infty$ for $\nu < 0$ and $\|h_0 \|x\|_r^\nu f(\tilde{x}))\| \to +\infty$ as $\|x\| \to \infty$ for $\nu > 0$. Since $\tilde{x} \neq 0$ then radial unboundedness of $\Lambda_r(\|x\|_r)$ imply radial unboundedness and properness of F.

III. To complete the proof using Proposition 3 (see below) it is sufficient to show that the function F is surjective on $\mathbb{R}^n$, *i.e.* for any $y \in \mathbb{R}^n$ there exists $x \in \mathbb{R}^n$ such that $F(x) = y$. Indeed, in this case the equation (3) with $h = h_0$ has a solution for any $x_i \in \mathbb{R}^n$. In order to prove this fact we use Theorem of Hadamard (see, *e.g.* Theorem 2.1. from [23]), which implies, under the condition (4) with $n \geq 2$, that the function F is surjective on $\mathbb{R}^n \backslash \{0\}$ (also on $\mathbb{R}^n$ due to $F(0) = 0$) and it is bijective for $n \geq 3$. ☐

Note that if $\nu > 0$ and the system (1) is unstable, then its trajectories demonstrate a finite-time escape phenomenon and the condition (4) my be invalid in such a case.

B. Relations between approximations obtained for different steps and initial conditions

The main result is as follows.

Proposition 3. *Let system (1) be* $\mathbf{r}$*–homogeneous with a degree ν. If $\{x_i\}_{i=0}^{\infty}$ is a sequence generated by (2) or (3) with the step h and the initial state x_0, then for any $\lambda > 0$, $y_i = \Lambda_r(\lambda)x_i$ is a sequence obtained by (2) or (3), respectively, with the step $\lambda^{-\nu}h$ and the initial state $y_0 = \Lambda_r(\lambda)x_0$.*

Proof. Fixing $\lambda > 0$ and multiplying both sides of (2) by the dilation matrix $\Lambda_r(\lambda)$ we obtain:

$$\begin{aligned} \Lambda_r(\lambda)x_{i+1} &= \Lambda_r(\lambda)x_i + h\Lambda_r(\lambda)f(x_i) \\ &= \Lambda_r(\lambda)x_i + h\lambda^{-\nu} f(\Lambda_r(\lambda)x_i), \end{aligned}$$

or, equivalently,

$$y_{i+1} = y_i + \lambda^{-\nu} h f(y_i)$$

that gives the desired result. The proof for the scheme (3) is the same. ☐

Note that y_i is an approximation of $X(\lambda^{-\nu}hi, y_0)$ for shifted instants of time. The following corollaries can be established.

Corollary 1. *Let system (1) be* $\mathbf{r}$*–homogeneous with a degree $\nu = 0$. Let for all $x_0 \in S_r(1)$ there exist sequences $\{x_i\}_{i=0}^{\infty}$ obtained by (2) or (3) with the step $h > 0$ and the initial state x_0 possessing one of the following properties:*

$$\sup_{i \geq 0} \|x_i\| < +\infty; \tag{5}$$

$$\lim_{i \to +\infty} x_i = 0. \tag{6}$$

Then for any $y_0 \in \mathbb{R}^n$ there exist sequences $\{y_i\}_{i=0}^{\infty}$ generated by (2) or (3) with the step h and the initial state y_0 possessing the same property.

Proof. There exists $\lambda > 0$ such that $y_0 = \Lambda_r(\lambda)x_0$ for some $x_0 \in S_r(1)$, next the result is a direct consequence of Proposition 3 since $y_i = \Lambda_r(\lambda)x_i$, $i \geq 0$. ☐

Corollary 2. *Let system (1) be* $\mathbf{r}$*–homogeneous with a degree $\nu \neq 0$. Let there exist $\rho_0 > 0$ and $h_0 > 0$ such that for any $x_0 \in S_r(\rho_0)$ the sequences $\{x_i\}_{i=0}^{\infty}$ obtained by (2) or (3) with the step h_0 and the initial state x_0 possess one of the properties (*(5)*), (*(6)*). Then for any $y_0 \in \mathbb{R}^n$ the sequences $\{y_i\}_{i=0}^{\infty}$ obtained by (2) or (3) with the step $h_0 \left(\frac{\|y_0\|_r}{\rho_0}\right)^{-\nu}$ and the initial state y_0 possess the same property.*

Proof. Note that for any $y_0 \in \mathbb{R}^n$ there exists $x_0 \in S_r(\rho_0)$ such that $y_0 = \Lambda_r(\lambda)x_0$ for $\lambda = \frac{\|y_0\|_r}{\rho_0}$. Let us take the sequence x_i obtained by (2) or (3) for the step h_0, then from Proposition 3, $y_i = \Lambda_r(\lambda)x_i$ is an approximation of solution of (1) with initial state y_0 obtained by (2) or (3) with the step $h_0 \left(\frac{\|y_0\|_r}{\rho_0}\right)^{-\nu}$. ☐

The results of corollaries 1 and 2 show advantages and limitations of the Euler method application for calculation of solutions of homogeneous systems with different degrees. For the case $\nu = 0$ the properties of approximation x_i depend on size of the step h, while for $\nu \neq 0$ if a scheme provides approximation of solutions for some h, then similar properties can be obtained for any initial condition with a properly scaled step h'.

Corollary 3. *Let system (1) be* $\mathbf{r}$*–homogeneous with a degree $\nu \neq 0$. Let for any $x_0 \in \mathbb{R}^n$ and some $h > 0$ there exist sequences $\{x_i\}_{i=0}^{\infty}$ obtained by (2) or (3) with initial state x_0 possessing one of the properties (*(5)*), (*(6)*). Then for any discretization step $h' > 0$ and for any $y_0 \in \mathbb{R}^n$ there exist sequences $\{y_i\}_{i=0}^{\infty}$ generated by (2) or (3) with the step h' and the initial state y_0 possessing the same property.*

Proof. There exist $\lambda > 0$ such that $h'\lambda^{-\nu} = h$, next for any $y_0 \in \mathbb{R}^n$, using the result of Proposition 3, we have that $y_i = \Lambda_r(\lambda)x_i$ with some $x_0 \in \mathbb{R}^n$. ☐

Thus, for $\nu \neq 0$ if a scheme provides approximation of solutions *globally* for some h, then similar properties can be obtained for *any* step h'. The latter is unlikely in general, thus using only homogeneity the global result for the case $\nu \neq 0$ cannot be obtained for (2) or (3).

V. Convergence of sequences $\{x_i\}_{i=0}^{\infty}$ generated by Euler methods

In this section we only study the stability features of $\{x_i\}_{i=0}^{\infty}$. The quality of the corresponding approximations of the continuous-time solutions $X(t, x_0)$ by $\{x_i\}_{i=0}^{\infty}$ is considered in the next section.

According to Theorem 1 (see [21], [22]), under Assumption 1 for the system (1) there is a twice continuously differentiable and

$\mathbf{r}$–homogeneous Lyapunov function $V : \mathbb{R}^n \to \mathbb{R}_+$ of degree $\mu > -\nu$ such that

$$\begin{aligned} a &= -\sup_{\xi\in S_r(1)} L_f V(\xi) > 0, \\ 0 < b &= \sup_{\xi\in B_r(1)} \left\| \frac{\partial V(\xi)}{\partial \xi} \right\| < +\infty, \\ c_1 &= \inf_{\xi\in S_r(1)} V(\xi),\ c_2 = \sup_{\xi\in S_r(1)} V(\xi), \\ c_1\|x\|_r^\mu &\le V(x) \le c_2\|x\|_r^\mu \quad \forall x \in \mathbb{R}^n. \end{aligned} \tag{7}$$

A. Convergence of explicit Euler scheme (2)

Let us take the discretization step $h > 0$ and consider the behavior of V on the sequence generated by (2). For this purpose define $x_i = \Lambda_r(\lambda)y_i$ with $y_i \in S_r(1)$ and $\lambda = \|x_i\|_r$:

$$\begin{aligned} V(x_{i+1}) - V(x_i) &= V(x_i + hf(x_i)) - V(x_i) \\ = \lambda^\mu[V(y_i + \lambda^\nu hf(y_i)) - V(y_i)] &= \lambda^{\nu+\mu} h \frac{\partial V(\xi)}{\partial \xi} f(y_i) \end{aligned}$$

for $\xi = y_i + \lambda^\nu \varrho f(y_i)$ with $\varrho \in [0, h]$ and the mean value theorem has been used on the last step. Note that

$$\underline{\sigma}(\|\xi\|_r) \le \|\xi\| \le \|y_i\| + \|x_i\|_r^\nu \varrho \|f(y_i)\| \le \overline{\sigma}(1) + g\|x_i\|_r^\nu h$$

for $g = \sup_{y\in S_r} \|f(y)\|$. Next,

$$\begin{aligned} V(x_{i+1}) - V(x_i) &= \lambda^{\nu+\mu} h\{\frac{\partial V(y_i)}{\partial y_i} f(y_i) \\ &+ \frac{\partial V(\xi)}{\partial \xi} f(y_i) - \frac{\partial V(y_i)}{\partial y_i} f(y_i)\} \\ &\le h\lambda^{\nu+\mu}\{-a + g\|\frac{\partial V(\xi)}{\partial \xi} - \frac{\partial V(y_i)}{\partial y_i}\|\}. \end{aligned}$$

Since $\|\frac{\partial V(\xi)}{\partial \xi} - \frac{\partial V(y_i)}{\partial y_i}\| \le k\|\xi - y_i\|$ where $k > 0$ is the Lipschitz constant of $\frac{\partial V(\xi)}{\partial \xi}$ on the set $B_r(\underline{\sigma}^{-1}[\overline{\sigma}(1) + g\|x_i\|_r^\nu h])$ (note that $\xi, y_i \in B_r(\underline{\sigma}^{-1}[\overline{\sigma}(1) + g\|x_i\|_r^\nu h])$), then

$$\begin{aligned} V(x_{i+1}) - V(x_i) &\le h\lambda^{\nu+\mu}\{-a + gk\|\xi - y_i\|\} \\ \le h\lambda^{\nu+\mu}\{-a + gk\lambda^\nu \varrho\|f(y_i)\|\} &\le h\lambda^{\nu+\mu}\{-a + g^2 k\lambda^\nu h\}. \end{aligned}$$

Therefore, the condition of convergence for (2) is

$$\lambda^\nu h < \frac{a}{g^2 k}, \tag{8}$$

where in the right-hand side all constants are independent on the discretization approach. If (8) is satisfied, then $V(x_{i+1}) < V(x_i)$, or $\|x_{i+1}\|_r < (c_1^{-1}c_2)^{1/\mu}\|x_i\|_r$. The following results can be easily obtained next.

Theorem 2. *Let Assumption 1 be satisfied with $\nu = 0$, then there exists the discretization step $h > 0$ such that the sequences $\{x_i\}_{i=0}^\infty$ obtained by (2) for any initial state $x_0 \in \mathbb{R}^n$ and the step h possess the following properties:*
(a) *$\|x_i\|_r < \gamma\|x_0\|_r$ for all $i \ge 0$ for some $\gamma \ge 1$;*
(b) $\lim_{i\to+\infty} x_i = 0$.

Proof. Take some $\rho > 0$ with $x_0 \in B_r(\rho)$, and select the Lipschitz gain k of $\frac{\partial V(\xi)}{\partial \xi}$ (V is twice continuously differentiable) onto the set $B_r(\eta_\rho)$ where $\eta_\rho = \max\{\underline{\sigma}^{-1}[\overline{\sigma}(1)+g], (c_1^{-1}c_2)^{1/\mu}\rho\}$, then for $h < \min\{1, ak^{-1}g^{-2}\}$ by consideration above we have $V(x_{i+1}) < V(x_i) < V(x_0)$ that implies $\|x_i\|_r < \gamma\|x_0\|_r$ (or, equivalently, $x_i \in B_r(\eta_\rho)$) for all $i = 0, 1, \ldots$ with $\gamma = (c_1^{-1}c_2)^{1/\mu}$. Therefore, for derived values of k and h the condition (8) is satisfied for all $i = 0, 1, \ldots$ and $\lim_{i\to+\infty} x_i = 0$. Global result for all initial conditions $x_0 \in \mathbb{R}^n$ follows from Corollary 1. □

Note that for $\nu = 0$ the discrete-time systems are homogeneous in the sense of [24].

Theorem 3. *Let Assumption 1 be satisfied with $\nu < 0$, then for any $\rho > 0$ there exists a discretization step $h_\rho > 0$ such that the sequences $\{x_i\}_{i=0}^\infty$ obtained by (2) for any initial state $x_0 \notin B_r(\rho)$ with a step $h \le h_\rho$ possess the following properties:*
(a) *$\|x_i\|_r < \gamma\|x_0\|_r$ for all $i \ge 0$ for some $\gamma \ge 1$;*
(b) *there exists $i_{x_0} > 0$ such that $x_{i_{x_0}} \in B_r(\rho)$.*

As follows from Theorem 3, in the case $\nu < 0$ for any $h > 0$ the explicit Euler scheme provides for the global convergence into some homogeneous ball $B_r(\rho)$, and $\rho \to 0$ as $h \to 0$. A similar result from [20] also states that the radius ρ is proportional to $h^{-1/\nu}$.

Proof. Take $\rho' > \rho > 0$ and consider $x_0 \in B_r(\rho') \setminus B_r(\rho)$. For a twice continuously differentiable and homogeneous Lyapunov functions V, which exists for the system (1) by Assumption 1 and satisfies (7), select the Lipschitz gain k of $\frac{\partial V(\xi)}{\partial \xi}$ onto the set $B_r(\eta_\rho)$ where $\eta_\rho = \max\{\underline{\sigma}^{-1}[\overline{\sigma}(1) + \rho^\nu g], (c_1^{-1}c_2)^{1/\mu}\rho'\}$, then for $h \le h_\rho < \min\{1, ak^{-1}g^{-2}\rho^{-\nu}\}$ by consideration above we have $V(x_{i+1}) < V(x_i) < V(x_0)$ while $x_i \in B_r(\rho') \setminus B_r(\rho)$, that implies $\|x_i\|_r < \gamma\|x_0\|_r$ with $\gamma = (c_1^{-1}c_2)^{1/\mu}$ (or, equivalently, $x_i \in B_r(\eta_\rho)$) for all such $i \ge 0$. Since the sequence of $V(x_i)$ is monotonously decreasing, then there is an index i_{x_0}such that $x_{i_{x_0}} \in B_r(\rho)$. These properties for all $x_0 \notin B_r(\rho)$ follow from the same arguments as used in Corollary 2. □

Theorem 4. *Let Assumption 1 be satisfied with $\nu > 0$, then for any $\rho > 0$ there exists a discretization step $h_\rho > 0$ such that the sequences $\{x_i\}_{i=0}^\infty$ obtained by (2) for any initial state $x_0 \in B_r(\rho)$ with a step $h \le h_\rho$ possess the following properties:*
(a) *$\|x_i\|_r < \gamma\|x_0\|_r$ for all $i \ge 0$ for some $\gamma \ge 1$;*
(b) $\lim_{i\to+\infty} x_i = 0$.

Proof. By Assumption 1 the system (1) admits a twice continuously differentiable and homogeneous Lyapunov functions V with the properties as in (7). Take a $\rho > 0$ with $x_0 \in B_r(\rho)$, and select the Lipschitz gain k of $\frac{\partial V(\xi)}{\partial \xi}$ onto the set $B_r(\eta_\rho)$ where $\eta_\rho = \max\{\underline{\sigma}^{-1}[\overline{\sigma}(1)+(c_1^{-1}c_2)^{\nu/\mu}\rho^\nu g], (c_1^{-1}c_2)^{1/\mu}\rho\}$, then for $h \le h_\rho < \min\{1, ak^{-1}g^{-2}(c_1^{-1}c_2)^{-\nu/\mu}\rho^{-\nu}\}$ by consideration above we have $V(x_{i+1}) < V(x_i) < V(x_0)$ that implies $\|x_i\|_r < \gamma\|x_0\|_r$ (or, equivalently, $x_i \in B_r(\eta_\rho)$) for all $i = 0, 1, \ldots$ with $\gamma = (c_1^{-1}c_2)^{1/\mu}$. Therefore, for derived value of k and $h \le h_\rho$ the condition (8) is satisfied for all $i = 0, 1, \ldots$ and $\lim_{i\to+\infty} x_i = 0$. □

As follows from Theorem 4, in the case $\nu > 0$ for any $h > 0$ the explicit Euler scheme provides for the asymptotic convergence to zero in some $B_r(\rho)$, and $\rho \to \infty$ as $h \to 0$. It can be shown that the radius ρ is proportional to $h^{-1/\nu}$.

B. Convergence of implicit Euler scheme (3)

Exactly the same results can be obtained for (3). Defining $x_{i+1} = \Lambda_r(\lambda)y_{i+1}$ with $y_{i+1} \in S_r(1)$ and $\lambda = \|x_{i+1}\|_r$, we obtain

$$\begin{aligned} V(x_{i+1}) - V(x_i) &= V(x_{i+1}) - V(x_{i+1} - hf(x_{i+1})) \\ = \lambda^\mu[V(y_{i+1}) - V(y_{i+1} - \lambda^\nu hf(y_{i+1}))] &= \lambda^{\nu+\mu} h\frac{\partial V(\xi)}{\partial \xi} f(y_{i+1}) \end{aligned}$$

for $\xi = y_{i+1} - \lambda^\nu \varrho f(y_{i+1})$ with $\varrho \in [0, h]$ after application of the mean value theorem on the last step. Next, similarly

$$\underline{\sigma}(\|\xi\|_r) \le \overline{\sigma}(1) + g\|x_{i+1}\|_r^\nu h,$$

and

$$V(x_{i+1}) - V(x_i) \le h\lambda^{\nu+\mu}\{-a + g\|\frac{\partial V(\xi)}{\partial \xi} - \frac{\partial V(y_{i+1})}{\partial y_{i+1}}\|\}.$$

Let $k > 0$ be the Lipschitz constant of $\frac{\partial V(\xi)}{\partial \xi}$ on the set $B_r(\underline{\sigma}^{-1}[\overline{\sigma}(1) + g\|x_{i+1}\|_r^\nu h])$, then as before

$$V(x_{i+1}) - V(x_i) \leq h\lambda^{\nu+\mu}\{-a + g^2 k\lambda^\nu h\},$$

and (8) is the condition of convergence for (3).

More advantageous conditions can be obtained by imposing some additional but mild restrictions (we also assume that solutions exists, *i.e.* the conditions of Proposition 2 are satisfied).

Theorem 5. *Let Assumption 1 hold, $V : \mathbb{R}^n \to \mathbb{R}_+$ be a continuously differentiable* $\mathbf{r}$*–homogeneous Lyapunov function of degree μ for the system* (1)*. Then for the sequence $\{x_i\}_{i=0}^\infty$ generated by the implicit scheme* (3) *with any step $h > 0$ and any $x_0 \in \mathbb{R}^n$, the sequence $\{V(x_i)\}_{i=1}^{+\infty}$ is monotonously decreasing to zero provided that*

$$\langle x - y, f(x)\rangle \neq \|x - y\| \cdot \|f(x)\| \tag{9}$$

for all $x \neq y$ such that $x, y \in \{z \in \mathbb{R}^n : V(z) = 1\}$.

It is easy to show that if the level set of the Lyapunov function V is convex, then the condition (9) of this theorem holds.

Proof. I. Let us show that $x_i \neq x_{i+1}$ if $x_i \neq 0$. Suppose to the contrary that $x_i = x_{i+1} \neq 0$. Then from (3) we have $f(x_{i+1}) = 0$. Global asymptotic stability of the system (1) implies that $f(x) \neq 0$ for $x \neq 0$, so $x_{i+1} = x_i = 0$ and it is a contradiction.

II. Since the Lyapunov function V is homogeneous then its level sets can be defined as follows $\Omega(\lambda) = \{\Lambda_r(\lambda^{1/\mu})x \in \mathbb{R}^n : V(x) \leq 1\}$, $\lambda > 0$. Moreover, the boundary of the level set coincides with the level surface, *i.e.* $\partial\Omega(\lambda) = \{\Lambda_r(\lambda^{1/\mu})x \in \mathbb{R}^n : V(x) = 1\}$. Let us consider the convex closure $\mathrm{co}(\partial\Omega(\lambda)) = \{z = \mu x + (1-\mu)y, \forall \mu \in [0, 1], \forall x, y \in \partial\Omega(\lambda)\}$. Obviously, $\Omega(\lambda) \subset \mathrm{co}(\partial\Omega(\lambda))$. So, for any $x \in \partial\Omega(\lambda)$ and any $z \in \Omega(\lambda)$ there exist $y \in \partial\Omega(\lambda)$ and $\alpha > 0$ such that $z = x + \alpha(y - x)$.

III. Let $\lambda_{i+1} = V(x_{i+1}) > 0$. Then $x_{i+1} \in \partial\Omega(\lambda_{i+1})$. Let us show that $x_i \notin \Omega(\lambda_{i+1})$. Toward a contradiction, suppose that in this case there exist $y \in \partial\Omega(\lambda_{i+1})$ and $\alpha > 0$ such that $x_i - x_{i+1} = \alpha(y - x_{i+1})$. For any $h > 0$ from (3) we obtain $hf(x_{i+1}) = \alpha(x_{i+1} - y)$. Under conditions of theorem we have that the vectors $x_{i+1} - y$ and $f(x_{i+1})$ cannot be co-directional for any $y \in \partial\Omega(\lambda_{i+1})$. We obtain a contradiction. Therefore, $x_i \notin \Omega(\lambda_{i+1})$ and $V(x_{i+1}) = \lambda_{i+1} < V(x_i)$.

IV. For any $x_0 \in \mathbb{R}^n$, since $V(x_i) \geq 0$ for any $i \geq 0$ and the sequence $\{V(x_i)\}_{i=1}^{+\infty}$ is monotonously decreasing, then there is $V_\infty \geq 0$ such that $\{V(x_i)\}_{i=1}^{+\infty}$ is asymptotically converging to V_∞. Assume that $V_\infty > 0$, then repeating the analysis performed on the previous step we obtain that there exists $\varepsilon_\infty > 0$ such that

$$V(y) - V(y - hf(y)) \leq -\varepsilon_\infty \tag{10}$$

for any $y \in \partial\Omega(V_\infty)$ (*i.e.* $V(y) = V_\infty$). Since $\{V(x_i)\}_{i=1}^{+\infty}$ is converging, then

$$\lim_{i\to+\infty} V(x_i) - V(x_i - hf(x_i)) = 0.$$

Since $V_\infty > 0$ the following new sequence is well defined for all $i \geq 0$:

$$y_i = \Lambda_r\left(\left(\frac{V_\infty}{V(x_i)}\right)^{1/\mu}\right)x_i$$

and $y_i \in \partial\Omega(V_\infty)$ (note that in the step I of the proof it has been established that $V(x_i) \neq 0$ for all $i \geq 0$). Thus, the inequality (10) is satisfied for $y = y_i$ for all $i \geq 0$ and $\lim_{i\to+\infty} x_i - y_i = 0$ since $\lim_{i\to+\infty} \frac{V_\infty}{V(x_i)} = 1$ for $V_\infty > 0$. Therefore, by continuity of V, there is an index $i_\infty \geq 0$ such that

$$V(x_i) = V(y_i) + \delta_i,\ V(x_i - hf(x_i)) = V(y_i - hf(y_i)) + \delta_i'$$

with $|\delta_i| \leq \frac{\varepsilon_\infty}{4}$ and $|\delta_i'| \leq \frac{\varepsilon_\infty}{4}$ for all $i \geq i_\infty$. Consequently, for all $i \geq i_\infty$:

$$\begin{aligned} V(x_i) - V(x_i - hf(x_i)) &= V(y_i) - V(y_i - hf(y_i)) + \delta_i + \delta_i' \\ &\leq -\varepsilon_\infty + \delta_i + \delta_i' \leq -\frac{\varepsilon_\infty}{2}, \end{aligned}$$

then

$$0 = \lim_{i\to+\infty} V(x_i) - V(x_i - hf(x_i)) \leq -\frac{\varepsilon_\infty}{2} < 0$$

which is a contradiction, and $V_\infty = 0$ with the sequence $\{V(x_i)\}_{i=1}^{+\infty}$ monotonously decreasing to zero. ☐

C. Divergence of explicit Euler scheme (2)

A consequence of corollaries 2 and 3 is that the Euler methods cannot be used for approximation of solutions of $\mathbf{r}$–homogeneous systems (1) for a degree $\nu > 0$ far outside and for $\nu < 0$ close to the origin. This fact also can be proven rigorously (see also [11]).

Theorem 6. *Let Assumption 1 be satisfied with $\nu \neq 0$, then for any $h > 0$ there exist $\rho_h > 0$ and $\gamma_h > 0$ such that $\|x_i\|_r > \gamma_h\|x_0\|_r$ for all $i \geq 0$ and all sequences $\{x_i\}_{i=0}^\infty$ obtained by (2) for $x_0 \in B_r(\rho_h) \setminus \{0\}$ if $\nu < 0$ and $x_0 \notin B_r(\rho_h)$ if $\nu > 0$.*

Proof. Under Assumption 1 the system (1) admits a twice continuously differentiable and homogeneous Lyapunov functions V with the properties as in (7) and

$$V(x_{i+1}) - V(x_i) = \lambda^\mu[V(y_i + \lambda^\nu hf(y_i)) - V(y_i)],$$

where $x_i = \Lambda_r(\lambda)y_i$ for some $y_i \in S_r(1)$ and $\lambda = \|x_i\|_r$ and x_{i+1} is calculated by (2). Note that $f(y_i) \neq 0$ for all $y_i \in S_r(1)$ from Assumption 1, then for any $h > 0$ there exist $\rho_h > 0$ such that

$$V(y_i + \lambda^\nu hf(y_i)) > \varsigma V(y_i),\ \varsigma > 1$$

for all $y_i \in S_r(1)$ and $\lambda \geq \rho_h$ (indeed, for any $y_i \in S_r(1)$ the vector $f(y_i)$ defines a fixed direction of movement for $y_i + \lambda^\nu hf(y_i)$, then $\inf_{y_i\in S_r(1)} V(y_i + \lambda^\nu hf(y_i)) > \varsigma \sup_{y_i\in S_r(1)} V(y_i)$ for a sufficiently high value of $\lambda^\nu h > 0$). If $\nu > 0$, then the same consideration can be repeated for all $i \geq 0$ under restriction $\rho_h > 1$ showing a divergence to infinity. For $\nu < 0$ it is necessary to impose $\rho_h < 1$. ☐

Remark 1. Note that in the proof of Theorem 6, for the case $\nu > 0$, a divergence to infinity is shown for any $h > 0$ for all $x_0 \notin B_r(\rho_h)$ with properly selected $\rho_h > 0$. The divergence rate can be estimated as follows for any $i \geq 0$:

$$\|x_{i+1}\|_r > \left(\varsigma\frac{c_1}{c_2}\right)^{1/\mu}\|x_i\|_r,$$

which is exponential if $\varsigma > \frac{c_2}{c_1} \geq 1$ (in general, a finite-time escape is possible for the original system). The result of this theorem can also be reformulated as that for any $\rho_h > 0$ there exist the discretization step $h > 0$ such that the trajectories diverge for $x_0 \notin B_r(\rho_h)$ if $\nu > 0$ and for $x_0 \in B_r(\rho_h) \setminus \{0\}$ if $\nu < 0$.

Theorems 3, 4 and 6 provide a complete characterization of properties of the explicit Euler method (2): in the case $\nu > 0$ the solutions of Explicit method diverge to infinity for sufficiently large initial conditions, whereas for $\nu < 0$ they do not asymptotically converge to zero. Theorem 5 shows that under (9) the implicit Euler method (3) can be always used.

VI. Absolute and Relative Errors of Discretized Homogeneous Systems

Standard characteristics of the discretization precision are studied in this section for homogeneous systems. To this end recall that $X(t,x_i), t > 0$ is a solution of (1) with the initial condition $x(0)=x_i$ and denote by $x_{i+1}(h,x_i)$ the value derived by (2) or (3) for the same x_i and $h>0$, then [25]

- **absolute error** is the magnitude of the difference between the exact value and its approximation:
$$\Delta(h,x_i)=\|X(h,x_i)-x_{i+1}(h,x_i)\|,$$
$$\Delta_r(h,x_i)=\|X(h,x_i)-x_{i+1}(h,x_i)\|_r;$$
- **relative error** expresses how large the absolute error is compared with the exact value:
$$\delta(h,x_i)=\tfrac{\Delta(h,x_i)}{\|X(h,x_i)\|},\ \delta_r(h,x_i)=\tfrac{\Delta_r(h,x_i)}{\|X(h,x_i)\|_r}.$$

The errors are given for two different norms, the conventional one $\|\cdot\|$ and the homogeneous norm $\|\cdot\|_r$, the former one is used habitually for evaluation of a discretization method precision, while the latter one suits better for analysis of homogeneous systems.

Proposition 4. *There exist $\underline{\varrho},\overline{\varrho}\in\mathcal{K}_\infty$ such that*
$$\underline{\varrho}(\Delta_r(h,x_i))\le\Delta(h,x_i)\le\overline{\varrho}(\Delta_r(h,x_i)),$$
and for $r_{\max}=1$:
$$n^{-\frac{1}{\rho}-\frac{1}{2}}\delta_r(h,x_i)\le\delta(h,x_i)\le n^{\frac{1}{\rho}+\frac{1}{2}}\delta_r(h,x_i).$$

Proof. Recall that there exist $\underline{\sigma},\overline{\sigma}\in\mathcal{K}_\infty$ such that
$$\underline{\sigma}(\|x\|_r)\le\|x\|\le\overline{\sigma}(\|x\|_r)\quad\forall x\in\mathbb{R}^n,$$
which in particular take the form $\forall x\in\mathbb{R}^n$:
$$\begin{cases} n^{-\frac{r_{\max}}{\rho}}\|x\|_r^{r_{\max}} & \text{if } \|x\|_r\le n^{\frac{1}{\rho}} \\ n^{-\frac{r_{\min}}{\rho}}\|x\|_r^{r_{\min}} & \text{if } \|x\|_r> n^{\frac{1}{\rho}}\end{cases}\le\|x\|$$
$$\le\sqrt{n}\begin{cases}\|x\|_r^{r_{\min}} & \text{if } \|x\|_r\le 1\\ \|x\|_r^{r_{\max}} & \text{if } \|x\|_r>1\end{cases},$$
then the result for the absolute error Δ follows with $\overline{\varrho}(s)=\underline{\sigma}(s)$ and $\overline{\varrho}(s)=\overline{\sigma}(s)$. For the relative error δ we obtain:
$$\frac{\underline{\varrho}(\Delta_r(h,x_i))}{\overline{\sigma}(\|X(h,x_i)\|_r)}\le\delta(h,x_i)\le\frac{\overline{\varrho}(\Delta_r(h,x_i))}{\underline{\sigma}(\|X(h,x_i)\|_r)}$$
and taking into account the specific selection for the functions $\underline{\sigma},\overline{\sigma}$ given above these relations can be rewritten as follows:
$$\delta_r(h,x_i)\frac{\begin{cases} n^{-\frac{r_{\max}}{\rho}}\Delta_r^{r_{\max}-1}(h,x_i) & \text{if } \Delta_r(h,x_i)\le n^{\frac{1}{\rho}}\\ n^{-\frac{1}{\rho}} & \text{if } \Delta_r(h,x_i)>n^{\frac{1}{\rho}}\end{cases}}{\sqrt{n}\begin{cases}1 & \text{if } \|X(h,x_i)\|_r\le 1\\ \|X(h,x_i)\|_r^{r_{\max}-1} & \text{if } \|X(h,x_i)\|_r>1\end{cases}}$$
$$\le\delta(h,x_i)$$
$$\le\delta_r(h,x_i)\frac{\sqrt{n}\begin{cases}1 & \text{if } \Delta_r(h,x_i)\le 1\\ \Delta_r^{r_{\max}-1}(h,x_i) & \text{if } \Delta_r(h,x_i)>1\end{cases}}{\begin{cases} n^{-\frac{r_{\max}}{\rho}}\|X(h,x_i)\|_r^{r_{\max}-1} & \text{if } \|X(h,x_i)\|_r\le n^{\frac{1}{\rho}}\\ n^{-\frac{1}{\rho}} & \text{if } \|X(h,x_i)\|_r>n^{\frac{1}{\rho}}\end{cases}}.$$
Let $r_{\max}=1$ then
$$n^{-\frac{1}{\rho}-\frac{1}{2}}\delta_r(h,x_i)\le\delta(h,x_i)\le n^{\frac{1}{\rho}+\frac{1}{2}}\delta_r(h,x_i).$$
□

The proven proposition (which is based on power asymptotics) guarantees that Δ_r and δ_r are equivalent characteristics of the conventional approximation errors Δ and δ.

Theorem 7. *Let the system (1) be $\mathbf{r}$–homogeneous of degree ν and $x_{i+1}(h,x_i)$ be calculated by the explicit* (2) *or implicit* (3) *Euler scheme for $x_i\in\mathbb{R}^n$ and $h>0$. Then*

1) $\Delta_r(h,\Lambda_r(\lambda)x_i)=\lambda\Delta_r(h\lambda^\nu,x_i)$ *and* $\delta_r(h,\Lambda_r(\lambda)x_i)=\delta_r(h\lambda^\nu,x_i)$ *for any $h>0$ and $x_i\neq 0$;*
2) $\delta_r(h,x_i)\to 0$ *as $x_i\to\infty$ if $\nu<0$;*
3) $\delta_r(h,x_i)\to 0$ *as $x_i\to 0$ if $\nu>0$.*

If, in addition, the system (1) *is asymptotically stable then*

4) $\delta_r(h,x_i)\to\infty$ *as $x_i\to 0$ if $-r_{\min}<\nu<0$;*
5) $\delta_r(h,x_i)\to\infty$ *as $x_i\to\infty$ if $\nu>0$.*

Proof. Let us provide a proof for the explicit Euler scheme (2), the implicit one (3) can be treated analogously.

1) Taking into account the identities $X(h,\Lambda_r(\lambda)x_i)=\Lambda_r(\lambda)X(h\lambda^\nu,x_i)$, $x_{i+1}(h,\Lambda_r(\lambda)x_i)=\Lambda_r(\lambda)x_{i+1}(h\lambda^\nu,x_i)$ (see Proposition 3) and $\|\Lambda_r(\lambda)y\|_r=\lambda\|y\|_r$ we immediately derive the required identities for absolute and relative errors.

2)-3) Since $\delta_r(0,x_i^*)=0$ for any $x_i^*\in S_r(1)$ then $\delta_r(h,\Lambda(\lambda)x_i^*)=\delta_r(h\lambda^\nu,x_i^*)\to 0$ as $\lambda\to+\infty$ if $\nu<0$ and as $\lambda\to 0$ if $\nu>0$.

4) First of all note that $\|f(x)\|_r\to+\infty$ as $\|x\|_r\to+\infty$ for $-r_{\min}<\nu$. This fact directly follows from $\mathbf{r}$–homogeneity: $f(x)=\|x\|_r^\nu\Lambda_r(\|x\|_r)f(x^*)$, where $x^*\in S_r(1)$. The system (1) is assumed to be asymptotically stable then for $\nu<0$ it is uniformly finite-time stable [26], [27], [28], [20]. Hence, there exists a neighborhood M_h of the origin, which is depended on $h>0$, such that $X(h,x_i)=0$ for all $x_i\in M_h$. Let us show that $\sup_{x_i\in B_r(\rho)\subset M_h}\delta(h,x_i)=+\infty$ for any $\rho>0$. Suppose to the contrary that there exists $\rho^*>0$ such that $\delta(h,x_i)<+\infty$ for all $x_i\in B_r(\rho^*)$. For boundedness of relative error the next identity is necessary: $x_{i+1}(h,x_i)=x_i+hf(x_i)=0$ for all $x_i\in B_r(\rho^*)\subset M_h$. In this case, $f(-x_i)=-f(x_i)$ and $f(hf(x_i))=-f(x_i)$. Since $x_i=\Lambda_r(\|x_i\|_r)x_i^*$ for some $x_i^*\in S_r(1)$, then $f(h\|x_i\|^\nu f(x_i^*))=-f(x_i^*)$ for $x_i\in B(\rho^*)\setminus\{0\}$. Hence, $\|f(h\|x_i\|^\nu f(x_i^*))\|$ remains bounded while $h\|x_i\|^\nu\|f(x_i^*)\|\to+\infty$ as $\|x_i\|\to 0$ (due to $\nu<0$). This contradicts to the fact proven above that $\|f(x)\|_r\to+\infty$ as $\|x\|_r\to+\infty$.

5) Let us show now that $\delta_r(h,\cdot)$ is unbounded in any neighborhood of infinity (*i.e.* in $U(\rho)=\{x_i\in\mathbb{R}^n:\|x_i\|>\rho\}$) if $\nu>0$. Asymptotic stability of $\mathbf{r}$–homogeneous system (1) with positive degree implies fixed-time convergence [29], [5], [30] to the unit ball independently of the initial condition, *i.e.* there exists $T^{\max}>0$ such that $\|X(t,x_i)\|_r<1$ for any $t>T^{\max}$ and any $x_i\in\mathbb{R}^n$. Then selecting $\lambda=(T^{\max}/h)^{1/\nu}$ we derive $\|X(\lambda^\nu h,x_i)\|_r<1$ for any $x_i\in\mathbb{R}^n$. Taking into account that $x_{i+1}(\lambda^\nu h,x_i)\to\infty$ as $x_i\to\infty$ we complete the proof. □

Therefore, the explicit and implicit Euler schemes, for any value of the discretization step, provide a good approximation (*i.e.* small relative error δ_r) of the system solutions if $\nu<0$ for big values of initial conditions, and if $\nu>0$ in a vicinity of the origin. Roughly speaking, if a homogeneous system has a slower rate of convergence than a linear one (far outside of the origin for $\nu<0$ or in a neighborhood of the origin for $\nu>0$), then the Euler methods ensure a good precision.

VII. Examples

Consider a planar benchmark:
$$\begin{aligned}\dot{x}_1 &= x_2,\\ \dot{x}_2 &= -k_1|x_1|^{1+2\nu}\mathrm{sign}(x_1)-k_2|x_2|^{\frac{1+2\nu}{1+\nu}}\mathrm{sign}(x_2),\end{aligned}$$
where $x=[x_1\ x_2]^{\mathrm{T}}\in\mathbb{R}^2$ is the state vector, $k_1>0$ and $k_2>0$ are the system parameters, $\nu>-0.5$ is the homogeneity degree for

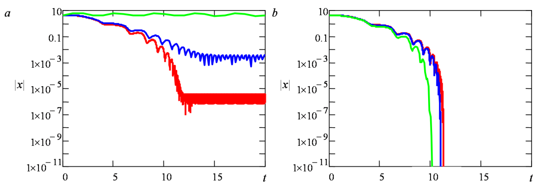

Figure 1. The results of simulation for $\nu = -0.25$

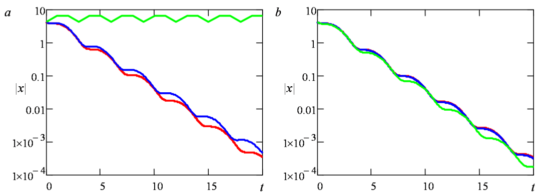

Figure 2. The results of simulation for $\nu = 0$

$r_i = 1 + (i-1)\nu$ and $i = 1, 2$. In [31] the following continuously differentiable Lyapunov function has been proposed for this system (the case of negative degree only has been analyzed in [31], but the same Lyapunov function can also be used for $\nu \geq 0$):

$$V(x) = c\left(k_1\varrho_2|x_1|^{\varrho_2^{-1}} + \frac{1}{2}x_2^2\right)^{\varrho_1} + k_1^{\varrho_2}x_1x_2,$$
$$c > 0,\ \varrho_1 = \frac{\nu+2}{2(\nu+1)},\ \varrho_2 = \frac{1}{2(\nu+1)},$$

which is positive definite with a negative definite derivative for

$$k_1 > k_2\varrho_2,\ c > \max\left\{\varrho_1^{-\varrho_1}, k_1^{\varrho_2}\frac{(2\nu+1)(k_2+1)+1}{2^{\nu\varrho_2}k_2(\nu+2)}\right\},$$

and these restrictions are accepted in the sequel. Implementation of (2) is straightforward, while for (3) we obtain:

$$\begin{aligned} x_{1,i+1} &= x_{1,i} + hx_{2,i+1}, \\ x_{2,i+1} &= x_{2,i} - h[k_1|x_{1,+1}|^{1+2\nu}\mathrm{sign}(x_{1,i+1}) \\ &\quad + k_2|x_{2,i+1}|^{\frac{1+2\nu}{1+\nu}}\mathrm{sign}(x_{2,i+1})] \\ &= x_{2,i} - h[k_1|x_{1,i} + hx_{2,i+1}|^{1+2\nu}\mathrm{sign}(x_{1,i} + hx_{2,i+1}) \\ &\quad + k_2|x_{2,i+1}|^{\frac{1+2\nu}{1+\nu}}\mathrm{sign}(x_{2,i+1})], \end{aligned}$$

and the last nonlinear equation can be solved with respect to $x_{2,i+1}$, then from the first equation the value of $x_{1,i+1}$ can be derived directly. In order to solve the last equation, the Secant method [3] is used in this work with a relative tolerance 10^{-11}.

Three values of degree ν are tested, -0.25, 0 and 0.25 (see figures 1, 2 and 3, respectively, where $|x(t)|$ is plotted in logarithmic scale), with $k_2 = 1$ and $k_1 = \frac{k_2}{\nu+1}$ (the condition $k_1 > k_2\varrho_2$ is satisfied). The methods (2) and (3) (part a of the figures for the explicit and part b for the implicit scheme, respectively) are applied with three values of discretization steps, 10^{-2}, 10^{-1} and 10^{0} (red, blue and green lines, respectively), for the same initial condition $x_0 = [3\ 3]^{\mathrm{T}}$.

The condition (9) for the method (3) is tested numerically, the corresponding levels of the Lyapunov function V are shown in Fig. 4, and it is easy to check that the level sets of the Lyapunov function V are convex. Hence, the conditions of Theorem 5 are satisfied and the implicit Euler scheme demonstrates global convergence of the approximations to the origin (up to the numeric precision used during simulations).

From the results presented in Fig. 1 we conclude that the explicit scheme (2) is converging for any $h > 0$ to a vicinity of the origin (theorems 3 and and 6) and for $h = 1$ the initial condition lies on the border of the attractor, while the implicit one (3) is converging globally (by Theorem 5). The system is finite-time stable in this case, and much better accuracy can be achieved with (3).

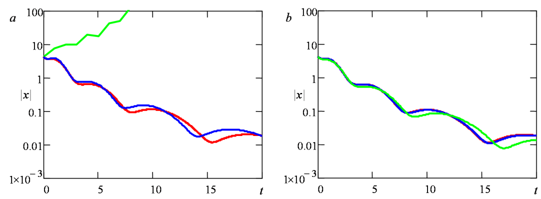

Figure 3. The results of simulation for $\nu = 0.25$

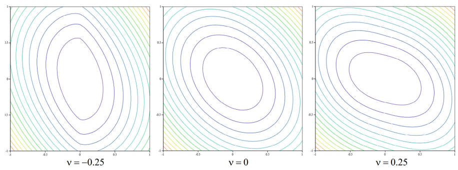

Figure 4. Contours of level sets for $V(x)$

In Fig. 2, the implicit scheme (3) is always converging (again confirming Theorem 5), but explicit method (2) is diverging for $h > 1$ for any initial conditions (an illustration of Corollary 1), and for $h = 1$ periodic sequences $\{x_i\}_{i=0}^{\infty}$ are generated globally. It is worth noting that the system is reduced to a linear one for $\nu = 0$.

For the case shown in Fig. 3, the explicit scheme (2) is converging for any $h > 0$ for sufficiently small initial conditions (theorems 4 and 6) and for $h = 1$ an unbounded trajectory is generated since the initial condition belongs to the domain of instability, while the implicit method (3) produces converging approximations.

From the results given in figures 1, 2 and 3, the implicit Euler scheme (3) demonstrates a faster rate of convergence for bigger values of the step h, while the explicit method (2) yields a better convergence speed for smaller values of the step h. Note that rate of convergence for $\{x_i\}_{i=0}^{\infty}$ is not related with the approximation accuracy of real solutions, which becomes better for $h \to 0$. Therefore, application of (3) with high values of h is not reasonable in the sense that the obtained sequence will converge to the origin faster than the real solution. This conclusion is illustrated in the parts b of figures 1, 2 and 3, where the sequences for $h = 0.1$ and $h = 0.01$ are very close to each other (and to the real solutions), while the sequences obtained for $h = 1$ evolve differently.

VIII. Conclusions

In this work a set of results has been obtained devoted to application of the explicit and implicit Euler methods for discretization of homogeneous systems. The main contributions can be summarized as follows:

- Basic properties deduced for implicit and explicit Euler methods by homogeneity for different values of *homogeneity degree* ν:
  - Homogeneity simplifies analysis of properties of the obtained discrete approximations of solutions, and there is a certain scalability between approximations calculated for different initial conditions and discretizations steps (Proposition 3).
  - For the case of $\nu = 0$ the properties of solution approximations are dependent on the discretization step value, and convergence to zero of the scheme for one value of the step does not imply the same property for another one (Theorem 2). However, verification of the global convergence can be performed on a sphere, if for the given discretization step

the approximation converges to the origin for all initial conditions on the sphere, then the convergence is preserved for any initial conditions (Corollary 1).

– For the case $\nu \neq 0$, convergence to the origin or boundedness of approximations obtained for some step on a sphere implies the same property for a properly selected discretization step for any initial condition (Corollary 2).

  * In the case of $\nu < 0$ it has been proved that the approximations *globally* converge to some vicinity of the origin (Theorem 3). The vicinity contracts to 0 as the step tends to 0.
  * For the case $\nu > 0$, it has been proved that for sufficiently small steps the approximations *locally* converge *in* some vicinity of the origin (Theorem 4). The vicinity covers all state space as the step tends to 0.

- For the case $\nu \neq 0$ global application of the explicit Euler scheme is troublesome since for any value of the step it becomes unstable for sufficiently small (with $\nu < 0$) or big (with $\nu > 0$) initial conditions (Theorem 6). Of course, for $\nu < 0$ convergence to a vicinity of the origin, which is shrinking when discretization step approaches zero [20], can be accepted in many applications, while global divergence for $\nu > 0$ should be strictly avoided.
- For the implicit Euler scheme it has been proved, under an additional mild condition, that solutions always exist for any initial conditions and discretization steps (Proposition 2). In addition, the approximations are converging to zero for any initial conditions and discretization steps if the level set of the Lyapunov function of the system is convex (Theorem 5). However, the implicit Euler method has a higher computational complexity than the explicit method, which is the price to pay for all its advantages. Thus, for $\nu < 0$ and for a sufficiently small discretization step the explicit Euler method can be a reliable choice.
- For any value of the discretization step, the implicit and explicit Euler methods provide a good approximation of the system solutions far outside of the origin for $\nu < 0$ and in a vicinity of the origin for $\nu > 0$ (Theorem 7).
- For $\nu < 0$ the explicit Euler method can be used outside of a vicinity of the origin and next switching to the implicit Euler methods is reasonable, in order to demonstrate convergence to the origin (initial application of the explicit method is motivated by its lower computational complexity).

Thus, application of the implicit Euler method to calculate global approximations of solutions of homogeneous systems with non-zero degree (having finite-time or fixed-time rates of convergence) is strongly recommended.

Future directions of research will include analysis of applicability of Euler methods for locally homogeneous systems, as well as analysis of properties of other methods for approximation of solutions of homogeneous systems and advantages of using a time-varying discretization step.

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